

# Modeling and Analysis of Laminated Composite Beams using Higher Order Theory

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**Abstract**— The present study deals with the assessment of higher order theory of laminated beams under static mechanical loads. The theory has been presented for general lay-up of the laminate. The displacement field is expressed in terms of only three primary displacement variables by satisfying exactly the conditions of zero transverse shear stress at the top and bottom and its continuity at layer interfaces. The governing equations of motion and boundary conditions are derived using virtual work. The number of primary displacement unknowns is three, which is independent of the number of layers and equal in number to the ones used in the first order shear deformation theory. Higher order theory thus preserves the computational advantage of an equivalent single layer theory. The Third order theory and First order shear deformation theory are assessed by comparison with the exact two-dimensional elasticity solution of the simply-supported beam. A theory is good only if it yields accurate results for all kinds of loads and for any lay-up of the beam. For this purpose, parametric studies for composite laminates and sandwich beams are conducted.

**Index Terms**—Composite Beams, First Order Shear Deformation Theory, Third Order Theory.

## I. INTRODUCTION

Laminated composite and sandwich beams are used extensively in various structures. For their efficient design, a good understanding of their deformation characteristics, stress distribution, natural frequencies and buckling loads under various load conditions are needed. A review of three-dimensional (3-D) continuum-based approaches, 2-D theories for plates and shells and 1-D theories for beams, along with their comparative study for plates under static loading, has been presented by Saravanas and Heyliger [1]. Analytical 2-D solutions for free vibration [2] are available only for simply-supported infinite flat panels and beams. The 3-D finite element analysis [3] results in large problem size which may become computationally costly for practical dynamics and control problems. Hence, accurate 1-D beam models are required without too much loss of accuracy compared to the 2-D models. Early works employed elastic beam models [4–6] with effective forces and moments due to induced strain of actuators. A discrete layer theory (DLT) with layerwise approximation of displacements was developed for elastic laminated beams with induced actuation strain by Robbins

and Reddy [7]. Classical laminate theory (CLT) [8], first order shear deformation theory (FSDT) [9] and the refined third order theory (TOT) [10,11] have been applied without electromechanical coupling to hybrid beams. The classical laminated beam theory (CLT), is an extension of the classical beam [13], theory. It neglects the effect of transverse shear strain. The CLT ignores the effect of transverse shear deformation and under predicts deflections and over predicts natural frequencies and buckling loads. The first order shear deformation theory (FSDT) based on Timoshenko's beam theory assumes linear variation of axial displacement  $u$  with the thickness coordinate  $z$ , across the thickness and the transverse displacement  $w$  to be independent of  $z$ . Thus the transverse shear strain in FSDT is constant across the thickness and the transverse shear stress is constant layerwise. Shear correction factor is therefore needed to rectify the unrealistic variation of the shear strain/stress through the thickness which actually even for an isotropic beam is approximated parabolic. Higher order shear deformation theories (HSDTs) involving higher order terms in the Taylor's expansion of the displacements in the thickness coordinate  $z$  have been developed for laminated composite and sandwich beams. The force resultants in FSDT have greater physical meaning than the ones used in higher order theories [14, 15] and the number of equations of motion is only three. Reddy derived a third order variationally consistent theory which satisfies the conditions of zero shear stress on the top and bottom surfaces of the beam. 1D models are used for the beams. In equivalent single layer (ESL) theories, global through-the-thickness approximations are used for the displacements as in CLT, FSDT, and HSDTs.

1D beam theories in which the displacement field is assumed to follow a global variation in the thickness direction across all the layers, independent of layup and material properties. The deflection  $w$  is usually assumed to be constant across the thickness and the inplane displacements are assumed to follow first the order, third order or higher order variations across the thickness. In view of the global variation of the displacements along the thickness direction, the continuity of transverse shear stress at the layer interfaces are not satisfied in these theories. The classical laminate theory, the first order shear deformation theory, refined third order theory and higher order theories belong to this class.

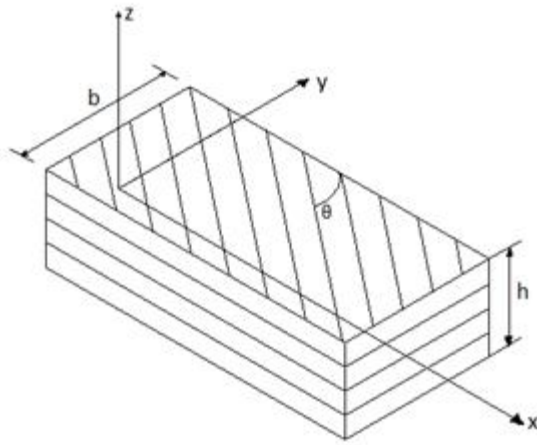


Figure 1. Geometry of composite beam

Shear correction factors are used in the FSDT, while it is not required in the higher order theories. Fig. 1 shows the geometry of composite beam.

## II. GOVERNING EQUATIONS FOR BEAMS

### A. First Order Shear Deformation Theory

The displacements, strains and stresses in FSDT are approximated as:

$$u = u_0(x, t) + z\psi_0(x, t) \quad (1)$$

$$w = w_0(x, t) \quad (2)$$

$$\varepsilon_x = u_{0,x} + z\psi_{0,x} \quad (3)$$

$$\gamma_{zx} = \psi_0 + w_{0,x} \quad (4)$$

$$\sigma_x = \hat{Q}_{11}\varepsilon_x = \hat{Q}_{11}(u_{0,x} + z\psi_{0,x}) \quad (5)$$

Without considering the nonlinear terms the nth Fourier component,

$$K\bar{U}^n = \bar{P}^n \quad (6)$$

Where,

$$\bar{U}^n = [u_0 \ w_0 \ \psi_0]^T_n,$$

$$\bar{P}^n = [P_1 \ P_2 \ P_3]^T_n$$

### B. Third Order Theory

In the third order theory the axial displacement is taken as:

$$u(x, z, t) = u_0(x, t) - zw_{0,xx} + (z - (4z^3/3h^2))\psi_x \quad (7)$$

The strain-displacement,

$$\varepsilon_x = u_0(x, t) - zw_{0,xx} + (z - (4z^3/3h^2))\psi_{x,x} \quad (8)$$

The stresses,

$$\sigma_x = \hat{Q}_{11}\varepsilon_x = \hat{Q}_{11}(u_0(x, t) - zw_{0,xx} + (z - (4z^3/3h^2))\psi_{x,x}) \quad (9)$$

Without considering the nonlinear terms the nth Fourier component,

$$K\bar{U}^n = \bar{P}^n \quad (10)$$

Where,

$$\bar{U}^n = [u_0 \ w_0 \ \psi_0]^T_n,$$

$$\bar{P}^n = [P_1 \ P_2 \ P_3]^T_n$$

## III. RESULTS

### A. Types of Beams

Two highly inhomogeneous simply-supported beams (a) and (b) are analysed. The stacking order is mentioned from the bottom. Beam (a) is composite beams of material 1 consisting of four plies of equal thickness  $0.25h$ . Beam (a) has symmetric lay-up  $[0^\circ/90^\circ/90^\circ/0^\circ]$ . The three layer sandwich beam (b) has graphite-epoxy faces and a soft core with thickness  $0.1h/0.8h/0.1h$ . The orientation  $\theta_k = 0$  for all the plies of beam (b). Beam (a), (b), are shown in Fig. 2 and Fig. 3 respectively.

The material properties are:

Material 1:

$$Y_1 = 181, Y_2 = Y_3 = 10.3 \text{ GPa}$$

$$G_{12} = G_{31} = 7.17, G_{23} = 2.87 \text{ GPa}$$

$$\nu_{12} = \nu_{13} = 0.25, \quad \nu_{23} = 0.33$$

Face:

$$Y_1 = 131.1, Y_2 = Y_3 = 6.9 \text{ MPa}$$

$$G_{12} = 3.588, G_{31} = 3.088, G_{23} = 2.3322 \text{ MPa}$$

$$\nu_{12} = \nu_{13} = 0.32, \quad \nu_{23} = 0.49$$

Core:

$$Y_1 = 0.2208, Y_2 = 0.2001, Y_3 = 2760 \text{ MPa}$$

$$G_{12} = 16.56, G_{31} = 545.1, G_{23} = 455.4 \text{ MPa}$$

$$\nu_{12} = 0.99, \nu_{13} = \nu_{23} = 3 \times 10^{-5}$$

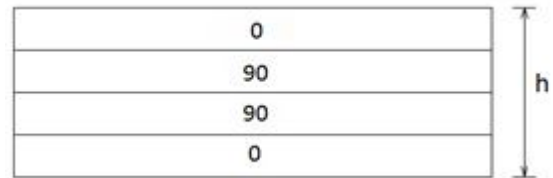


Figure 2. Beam (a)



Figure 3. Beam (b)

### B. Types of Load

Consider the following types of load for the analysis:

1. Sinusoidal load:  $p_z^2 = p_0 \sin(\pi x/a)$ .
2. Uniformly distributed load:  $p_z^2 = p_0$ . Its non-zero Fourier components are  $(p_z^2)_n = 4p_0/n\pi$  for n odd.

### C. Static Response

Static response is compared for two kinds of load applied

TABLE I. EXACT RESULTS FOR STATIC SINUSOIDAL LOAD AND FOR FSDT AND TOT

	Beam	a			B		
	S	5	10	20	5	10	20
$\bar{w}(0.5a, 0)$	FSDT	-1.6999	-1.0144	-0.8431	-5.9385	-2.9557	-2.2100
	TOT	-1.9371	-1.0754	-0.8584	-5.5165	-2.3680	-1.5777
	Exact	-2.1393	-1.1397	-0.8816	-6.3471	-2.5978	-1.6469
$\bar{\sigma}_x(0.5a, 0.5h)$	FSDT	0.6312	0.6312	0.6312	1.1410	1.1410	1.1410
	TOT	0.7119	0.6510	0.6360	1.3139	1.1834	1.1510
	Exact	0.8912	0.7435	0.7030	1.6580	1.3519	1.2725
$\bar{\sigma}_x(0.5a, -0.5h)$	FSDT	-0.6312	-0.6312	-0.6312	-1.1410	-1.1410	-1.1410
	TOT	-0.7119	-0.6510	-0.6360	-1.3139	-1.1834	-1.1510
	Exact	-0.8946	-0.7415	-0.7023	-1.6738	-1.3529	-1.2725

TABLE II. EXACT RESULTS FOR STATIC UNIFORMLY DISTRIBUTED LOAD AND FOR FSDT AND TOT

	Beam	a			B		
	S	5	10	20	5	10	20
$\bar{w}(0.5a, 0)$	FSDT	-2.2391	-1.3158	-1.0849	-5.2737	-2.5823	-1.9095
	TOT	-2.5530	-1.3975	-1.1056	-7.3339	-3.1042	-2.0404
	Exact	-2.6748	-1.4343	-1.1152	-7.8930	-3.2521	-2.0778
$\bar{\sigma}_x(0.5a, 0.5h)$	FSDT	0.8502	0.8502	0.8502	1.5369	1.5369	1.5369
	TOT	1.0097	0.8901	0.8602	1.8815	1.6230	1.5584
	Exact	1.0711	0.9058	0.8641	1.9655	1.6441	1.5637
$\bar{\sigma}_x(0.5a, -0.5h)$	FSDT	-0.8502	-0.8502	-0.8502	-1.5369	-1.5369	-1.5369
	TOT	-1.0097	-0.8901	-0.8602	-1.8815	-1.6230	-1.5584
	Exact	-1.0602	-0.9031	-0.8634	-1.9664	-1.6443	-1.5637

TABLE III. % ERROR FOR FSDT AND TOT FOR STATIC SINUSOIDAL LOAD

	Beam	a			B		
	S	5	10	20	5	10	20
$\bar{w}(0.5a, 0)$	FSDT	-20.54	-10.99	-4.37	-6.44	13.78	34.18
	TOT	-9.45	-5.64	-2.63	-13.08	-8.84	-4.20
$\bar{\sigma}_x(0.5a, 0.5h)$	FSDT	-29.16	-15.09	-10.21	-31.18	-15.60	-10.33
	TOT	-20.11	-12.43	-9.53	-20.75	-12.46	-9.55
$\bar{\sigma}_x(0.5a, -0.5h)$	FSDT	-29.44	-14.86	-10.12	-31.83	-15.66	-10.33
	TOT	-20.42	-12.19	-9.44	-21.50	-12.53	-9.55

on the top surface of the beam a sinusoidal load  $p_0 \sin(\pi x/a)$ , a uniformly distributed load  $p_0$ . For a thick beam with  $S = 5$  and for a thin beam with  $S = 100$ . Very good convergence of the central deflection and stresses is obtained by taking Fourier terms from  $n = 1$  to  $n = N = 159$ , i.e., for 80 odd terms in the Fourier series expansion. The deflection and stresses are non-dimensionalised as follows with  $Y_0 = 6.9$  and  $10.3$  GPa.

$$\bar{w} = 100wY_0/hS^4p_0, \quad \bar{\sigma}_x = \sigma_x/S^2p_0,$$

The central deflection  $\bar{w}(0.5a, 0)$ , axial stresses  $\bar{\sigma}_x(0.5a, 0.5h)$  and  $(0.5a, -0.5h)$  for the beams (a), (b) under sinusoidal static load, obtained by the FSDT and TOT and compare with the 2D exact solution, are listed in Table I. for  $S = 5$  (thick beam),  $S = 10$  (moderately thick beam) and  $S = 20$  (thin beam). Similar results for uniformly distributed load are presented in Table II. Percentage error for FSDT and TOT for static sinusoidal load and for static uniformly distributed load are listed in Table III and Table IV. Fig. 4 to Fig. 11 shows that percentage error varies with respect to  $h/a$  for deflection and stress.



TABLE IV. % ERROR FOR FSDT AND TOT FOR STATIC UNIFORMLY DISTRIBUTED LOAD

	Beam	a			B		
	S	5	10	20	5	10	20
$\bar{w}(0.5a, 0)$	FSDT	-16.29	-8.26	-2.71	-33.18	-20.60	-8.10
	TOT	-4.55	-2.57	-0.86	-7.08	-4.55	-1.80
$\bar{\sigma}_x(0.5a, 0.5h)$	FSDT	-20.62	-6.14	-1.61	-21.81	-6.52	-1.72
	TOT	-5.73	-1.74	-0.46	-4.27	-1.28	-0.34
$\bar{\sigma}_x(0.5a, -0.5h)$	FSDT	-19.80	-5.86	-1.53	-21.84	-6.53	-1.72
	TOT	-4.76	-1.44	-0.38	-4.32	-1.29	-0.34

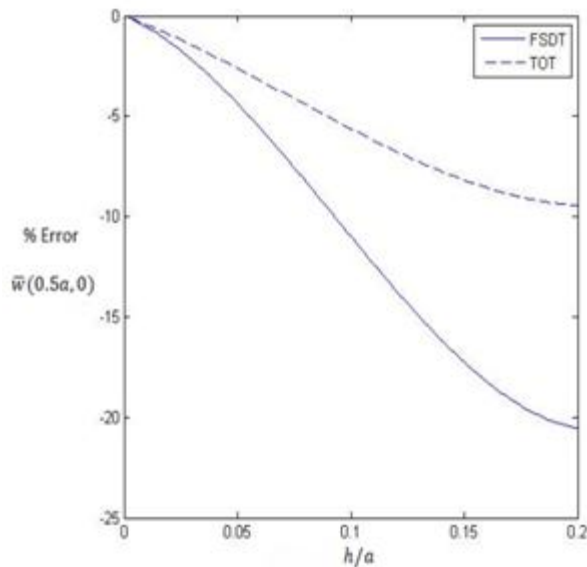


Figure 4. % Error for deflection for test beam (a) under static sinusoidal load

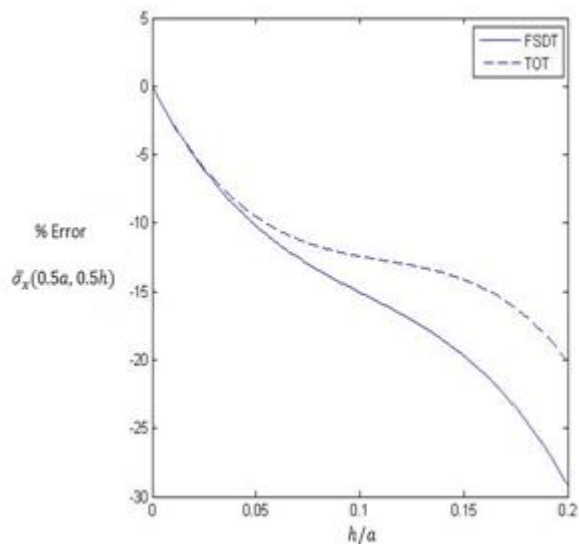


Figure 5. % Error for stress for test beam (a) under static sinusoidal load

#### IV. EFFECT OF ANGLES

##### A. Symmetric Laminates

When ply stacking sequence, material, and geometry (i.e. ply thicknesses) are symmetric about the midplane of the

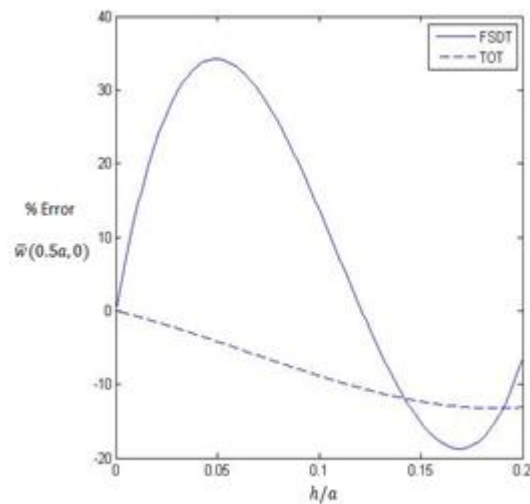


Figure 6. % Error for deflection for test beam (b) under static sinusoidal load

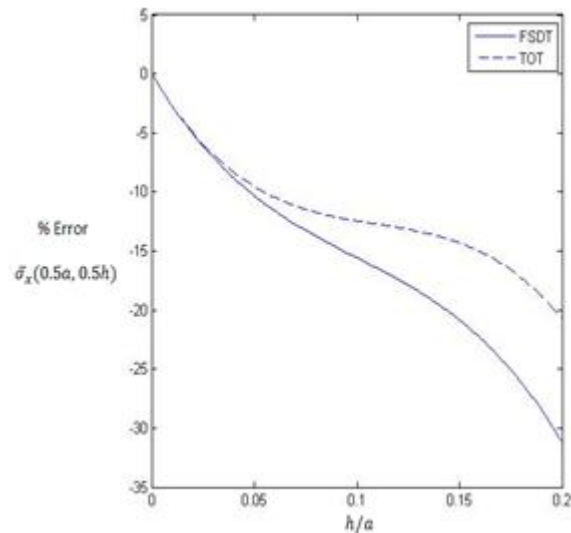


Figure 7. % Error for stress for test beam (b) under static sinusoidal load

laminates, the laminate is called the symmetric laminate. For a symmetric laminate, the upper half through the laminate thickness is a mirror image of the lower half. The laminates  $(-45/45/45/-45) = (-45/45)_s$  and  $(0/90/90/0) = (0/90)_s$ , with all layers having the same thickness and material, are example of a symmetric cross ply laminate. Now, consider the following symmetric laminates with angles  $(0/0)_s$ ,  $(0/15)_s$ ,  $(0/30)_s$ ,

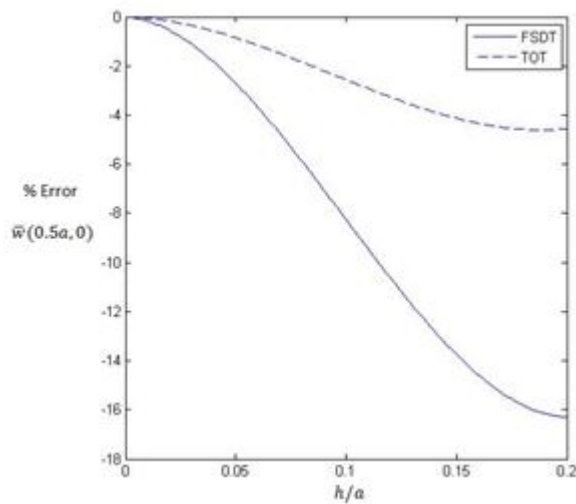


Figure 8. % Error for deflection for test beam (a) under static uniformly distributed load

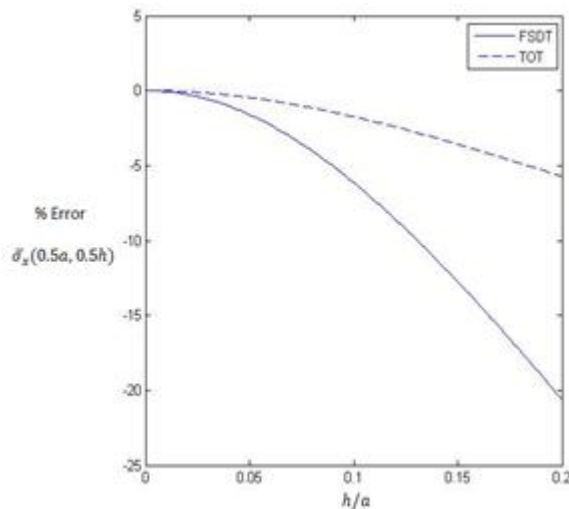


Figure 9. % Error for stress for test beam (a) under static uniformly distributed load

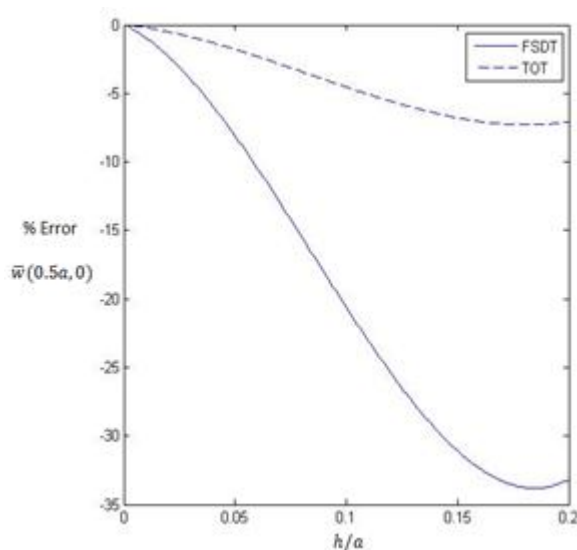


Figure 10. % Error for deflection for test beam (b) under static uniformly distributed load

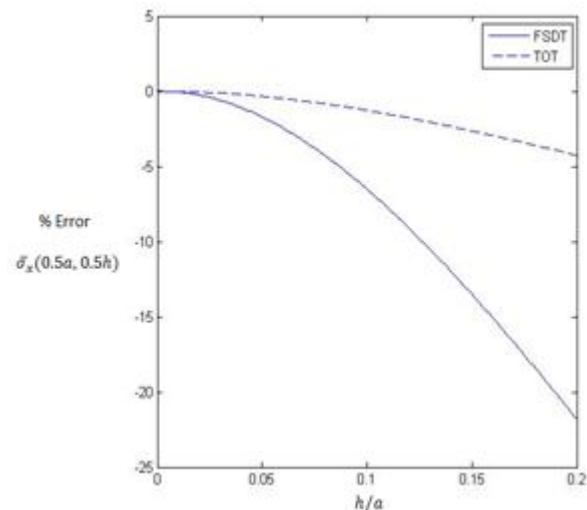


Figure 11. % Error for stress for test beam (b) under static uniformly distributed load

Stress for FSDT and TOT, for  $S = 5$  and for  $S = 10$ , for static sinusoidal load and for static uniformly distributed load are listed in Table V and Table VI. Figs. 12 to 19 shows that effect of angles on deflection and stress for static sinusoidal load and for uniformly distributed load.

## CONCLUSIONS

A theory is good only if it yields accurate results for all kinds of loads and for any lay-up of the beam. Now we can compare the results obtained by the FSDT and TOT for the static sinusoidal load and the static uniformly distributed load are presented in Tables I. and II. For the sinusoidal case, the percentage error in deflection and stresses for the thickness parameter ( $S = a/h$ ),  $S = 5$  (thick beam),  $S = 10$  (moderately thick beam), and  $S = 20$  (thin beam), by TOT less than the percentage error obtained by FSDT for both beam (a), (b). For the static uniformly distributed load case, the percentage error in deflection and stresses for  $S = 5$  (thick beam),  $S = 10$  (moderately thick beam) and  $S = 20$  (thin beam) by TOT less than the percentage error obtained by FSDT for both beam (a), (b). The TOT yields much more accurate results than the FSDT. For the symmetric laminates the effect of angles on deflections and stresses for FSDT and TOT for static sinusoidal load and for static uniformly distributed load are presented in Tables V. and VI. For the sinusoidal case, the deflections and stresses for  $S = 5$  and  $S = 10$ , are decreases as the angle increases and for the static uniformly distributed load case, the deflections and stresses for  $S = 5$  and  $S = 10$ , are also decreases as the angle increases and in both the load cases TOT gives much more accurate results than the FSDT.

## REFERENCES

- [1] D.A. Saravananos, P.R. Heyliger, Mechanics and computational models for laminated piezoelectric beams, plates and shells, Applied Mechanics Review 52 (1999) 305–320.
- [2] P.R. Heyliger, S.B. Brooks, Exact free vibration of

$(0/45)_s$ , and  $(0/60)_s$ . Effect of angles on Deflection and

- piezoelectric laminates in cylindrical bending, *International Journal of Solids and Structures* 32 (1995) 2945–2960.
- [3] S.K. Ha, C. Keilers, F.K. Chang, Three-dimensional finite element analysis of composite structures containing distributed piezoceramic sensors and actuators, *American Institute of Aeronautics and Astronautics Journal* 30 (1992) 772–780.
- [4] T. Bailey, J.E. Hubbard, Distributed piezoelectric–polymer active vibration control of a cantilever beam, *Journal of Guidance, Control and Dynamics* 8 (1985) 605–611.
- [5] S. Im, S.N. Atluri, Effects of a piezo-actuator on a finitely deformed beam subject to general loading, *American Institute of Aeronautics and Astronautics Journal* 27 (1989) 1801–1807.
- [6] R. Chandra, I. Chopra, Structural modelling of composite beams with induced-strain actuators, *American Institute of Aeronautics and Astronautics Journal* 31 (1993) 1692–1701.
- [7] D.H. Robbins, J.N. Reddy, Analysis of piezoelectrically actuated beams using a layer-wise displacement theory, *Computers and Structures* 41 (1991) 265–279.
- [8] H.S. Tzou, Distributed sensing and controls of flexible plates and shells using distributed piezoelectric element, *Journal of Wave Material Interaction* 4 (1989) 11–29.
- [9] X.D. Zhang, C.T. Sun, Formulation of an adaptive sandwich beam, *Smart Materials and Structures* 5 (1996) 814–823.
- [10] K. Chandrashekhara, P. Donthireddy, Vibration suppression of composite beams with piezoelectric devices using a higher order theory, *European Journal of Mechanics A/Solids* 16 (1997) 709–721.
- [11] X.Q. Peng, K.Y. Lam, G.R. Liu, Active vibration control of composite beams with piezoelectrics: a finite element model with third order theory, *Journal of Sound and Vibration* 209 (1998) 635–650.
- [12] Alam N., Upadhyay N.K., Finite Element Analysis of Laminated Composite Beams for Zigzag theory using MatLab, *International Journal of Mechanics and Solids*, 5, 1–14, 2010.
- [13] Averill, R. C., Static and dynamic response of moderately thick laminated beams with damage, *Composite Engg.*, 1994, 4, 381–395.
- [14] Cho M, Paramerter R.R., Efficient higher order plate theory for laminated composites, *Compos Struct* 1992, 20(2), 113–23.
- [15] Kumari P., Nath J. K., Kapuria S, Dumir. P. C., An improved third order theory and assessment of efficient zigzag theory for angle-ply flat hybrid panels, *Composite Structures* 83 (2008) 226–236.